Lakehead University

Honours Seminar

Cantor Minimal Systems and the Bratteli-Vershik Model

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Abstract

In topological dynamics, the class of Cantor minimal systems is of great interest. These are topological dynamical systems on a totally disconnected metric space without isolated points, where every point has dense orbit. The dimension group K^0 classifies these systems under orbit equivalence and strong orbit equivalence up to isomorphism. We introduce Bratteli diagrams and Bratteli-Vershik systems as a combinatorial model for AF-algebras. Every Cantor minimal system is conjugate to a Bratteli-Vershik system. We conclude by discussing the real structures that arise from these Bratteli diagrams.

1 Cantor Minimal Systems

The Cantor set is a famous example in analysis and topology because it highlights key concepts but is also peculiar. It bears of the name of Georg Cantor (1845–1918), but was first introduced by Henry John Stephen Smith (1826–1883). In an 1875 paper, Smith defined a version of the Cantor set as an example of an integrable function with infinitely many discontinuities. It was later defined by Georg Cantor in his 1883 paper titled *On infinite linear manifolds of points, Part 5* [18]. In this chapter, we define the Cantor set and discuss its properties. We then equip the Cantor set with a self-homeomorphism to form a Cantor minimal system. These are the dynamical systems of interest in this thesis.

1.1 The Cantor Set

First, we will introduce the Cantor Set using the ternary set construction. Consider the closed interval $\mathcal{C}_0 = [0, 1]$. Define $\mathcal{C}_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $\mathcal{C}_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. For n ≥ 2, construct $\mathcal{C}_{n+1}$ by removing the open middle third from each of the disjoint closed intervals of $\mathcal{C}_n$. It follows that $\mathcal{C}_n$ is the union of 2^n disjoint closed intervals of length $(\frac{1}{3})^n$.

Definition 1.1. Let $\mathcal{C}_n$ be constructed as above. The Cantor set $\mathcal{C}$ is defined to be $\mathcal{C} = \bigcap_{n=0}^{\infty} \mathcal{C}_n$.

It is easy to see that $\mathcal{C}$ is non-empty since it contains 0 and 1. In fact, $\mathcal{C}$ contains the endpoints of each subinterval in $\mathcal{C}_n$, so it is infinite. We can even go a step further and say that $\mathcal{C}$ is uncountable.

Theorem 1.2. The Cantor set is uncountable.

Sketch of the proof. Since $[0, 1]$ is uncountable, we prove this by constructing a surjective function $f : \mathcal{C} \to [0, 1]$. If we write each number in $[0, 1]$ in ternary form (ie. as a decimal consisting only of the digits $\{0, 1, 2\}$), a number is in the Cantor set if and only if its ternary representation is comprised entirely of 0s and 2s. We define f to take an element of the Cantor set in its ternary expansion and replace all the 2s with 1s. The range of f is the set $[0, 1]$ with each element in binary. Thus, f is surjective, which shows that $\mathcal{C}$ is uncountable [14].

We will now define and prove the properties of $\mathcal{C}$ which make it a unique and interesting example in analysis and topology.

Definition 1.3. Let S be a set. We say that S is *compact* if every open cover of S has a finite subcover.

Example 1.4. Every finite set is compact.

Example 1.5. The set $U = \{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ is an open cover of (0,1). If $\{U_n : n = 1, 2, ...k\}$ is a finite subcover of U, $\frac{1}{k+1} \notin \{U_n\}$. Thus $(0, 1)$ is not compact.

From the *Heine-Borel Theorem*, we know a set in $\mathbb{R}^k$ is compact if and only if it is closed and bounded. Since $\mathcal{C} \subset \mathbb{R}$, we can easily show that $\mathcal{C}$ is compact.

Theorem 1.6. The Cantor set is compact.

Proof. Since $\mathcal{C}$ is the intersection of closed sets, it is closed, and since $\mathcal{C}$ is contained in $[0, 1]$, it is bounded. Hence, by the Heine-Borel Theorem, $\mathcal{C}$ is compact. □

Next, we show that the Cantor set is totally disconnected.

Definition 1.7. We say that a set S is not connected if there are non-empty sets A, B, with $S = A \cup B$, and $A \cap \overline{B} = B \cap \overline{A} = \emptyset$. S is *connected* if it is not not connected.

Example 1.8. All intervals in $\mathbb{R}$ are connected (that is, sets of the form (a, b), $(a, b]$, $[a, b)$, and $[a, b]$ with $a < b$).

Definition 1.9. Let S be a set and $p \in S$. The *connected component* of the element p, denoted $[p]$, is the union of the collection of the connected subsets of S that contain p.

Definition 1.10. Let S be a set. We say S is *totally disconnected* if for every $p \in S$, we have $[p] = \{p\}$.

Theorem 1.11. The Cantor set is totally disconnected.

Proof. Suppose $p \in \mathcal{C}$. Since $\mathcal{C}_n$ consists of intervals of length $(\frac{1}{3})^n$, and $n \to \infty$, $\mathcal{C}$ has no intervals of positive length. Since the connected subsets of $\mathbb{R}$ are exactly the intervals, the only connected subset of $\mathcal{C}$ is $\{p\}$. □

The last, and probably least intuitive property of the Cantor set that we will discuss is the fact that it has no isolated points.

Definition 1.12. Let (X, d) be a metric space, and $S \subseteq X$. An *isolated point* of S is a point $p \in S$ such that there exists a neighbourhood A of p with $S \cap A = \{p\}$.

Theorem 1.13. The Cantor set has no isolated points.

Proof. Suppose $x \in \mathcal{C}$ and let A be a neighbourhood of x. Equip $\mathbb{R}$ with the usual metric. We must show that A contains at least one other point of $\mathcal{C}$. Let $B(x, r) \subseteq A$ be an open ball centred at x with radius r. We can choose a sufficiently large n so that $(\frac{1}{3})^n < r$. Since $x \in \mathcal{C}$, $x \in \mathcal{C}_n$, and so x is an element of one of the closed, disjoint intervals of length $(\frac{1}{3})^n$ that make up $\mathcal{C}_n$. Call this interval $I = [a, b]$. Since $x \in I$ and $(\frac{1}{3})^n < r$, it follows that $I \subset B(x, r)$. But, there are at least two other points of $\mathcal{C}$ in I, namely, a and b (there is only one other point if either $x = a$ or $x = b$). This shows that x is not an isolated point of $\mathcal{C}$. Since x was arbitrary, $\mathcal{C}$ has no isolated points. □

Remark 1.14. Sets that are closed with no isolated points are called *perfect* sets, so the Cantor set is perfect.

1.2 Cantor Minimal Systems

Now that we are familiar with the properties of the usual Cantor set, we can generalize these properties to the dynamical systems of interest, Cantor minimal systems.

Definition 1.15. A *topological dynamical system* is a pair (X, T) where X is a compact metrizable space and T a self-homeomorphism.

Definition 1.16. A *Cantor system* is a topological dynamical system (X, T) where X is totally disconnected and has no isolated points.

Example 1.17. [9] Let $\mathfrak{L} = \{1, 2, ..., l\}$ be a finite alphabet and let $\Omega = \mathfrak{L}^{\mathbb{Z}}$ be the compact space of functions from $\mathbb{Z}$ to $\mathfrak{L}$. One can think of this as the space of bi-infinite sequences on $\mathfrak{L}$. Let $S : \Omega \to \Omega$ be the left shift map: $S\omega_n = \omega_{n-1}$. For any S-invariant closed subset $X \subseteq \Omega$, (X, S) is a Cantor system.

Definition 1.18. Let (X, T) be a Cantor system. The set $\mathcal{O}_T(x) = \{T^n(x) | n \in \mathbb{Z}\}$ is called the *orbit* of the point $x \in X$ under T.

Definition 1.19. Let (X, T) be a Cantor system. A point $x \in X$ is said to have *dense orbit* if $\overline{\mathcal{O}_T(x)} = X$, where $\overline{\mathcal{O}_T(x)}$ denotes the closure of $\mathcal{O}_T(x)$.

Definition 1.20. [11] A homeomorphism $T : X \to X$ is said to be *minimal* if the orbit of every point $x \in X$ is dense in X.

Example 1.21. [2] Let X be the unit circle in $\mathbb{R}^2$ and $\phi : X \to X$ be an irrational rotation. The orbits generated by the action of ϕ on $\mathbb{Z}$ are dense, and so (X, ϕ) is a minimal system.

Definition 1.22. A *Cantor minimal system* is a Cantor system (X, T) with the added property of minimality.

Example 1.23. [2] The odometer, or adding machine, is a classic example of a Cantor minimal system. It is defined as follows. Let $X = \{0, 1\}^{\mathbb{N}}$, ie. the elements of X are countable sequences of 0s and 1s. Defining $D : X \times X \to \mathbb{R}$ as $d(f, g) = \Sigma_{n=0}^{\infty} 2(f(n) - g(n))/3^n$ makes X into a metric space. Define $\varphi : X \to X$ as $x \mapsto x + (1, 0, 0, 0, ...)$ with carry over to the right. This is a homeomorphism. For instance,

$$\varphi(1, 1, 0, 1, 0, 1, ...) = (0, 0, 1, 1, 0, 1, ...)$$

φ also wraps around so that $\varphi(1, 1, 1, 1, ...) = (0, 0, 0, 0, ...)$.

2 C*-Algebras and the K^0 Group

In this chapter, we will introduce C*-algebras and projection operators. We form the semigroup $\mathcal{D}(A)$ by introducing the notion of Murray-von Neumann equivalence and generalizing it for different sized projection matrices. We then form the K^0 group by completing the semigroup S to form a group via the Grothendieck construction, and provide a concrete picture of K^0. We will then see how K^0 provides us with a nice classification for Cantor minimal systems. In particular, in [7] it was proved that for Cantor minimal systems, orbit equivalence and strong orbit equivalence are invariant under K^0.

2.1 C*-Algebras and Real Structures

Definition 2.1. [13] An *algebra* is a vector space with a bilinear map

$$A^2 \to A,\ (a,b) \to ab$$

such that

$$a(bc) = (ab)c$$

for all $a, b, c \in A$.

Example 2.2. [13] The set of all bounded complex valued functions on a set S is an algebra when equipped with the sup norm

$$\|f\|_\infty = sup_{s\in S}|f(x)|$$

and pointwise operations

$$(fg)(x) = f(x) + g(x)$$
$$(f+g)(x) = f(x)g(x)$$
$$(\lambda f)(x) = \lambda f(x).$$

In fact, this algebra is complete, making it a *Banach algebra*.

Definition 2.3. A complex C*-algebra (or just *C*-algebra*) is a complete algebra A over the field $\mathbb{C}$ with an operation $* : A \to A$ with the following properties:

$$(\lambda a + b)^* = \bar{\lambda} a^* + b^*$$
$$(ab)^* = b^* a^*$$
$$a^{**} = a$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$, and a norm $\|\cdot\| : A \to \mathbb{R}$ satisfying

$$\|ab\| \leq \|a\| \, \|b\|$$

and the C*-identity

$$\|a^*a\| = \|a\|^2.$$

Example 2.4. $M_n(\mathbb{C})$ equipped with the norm $\|A\| = sup\{\|Ax\| \mid \|x\| \leq 1\}$ and $* : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ the conjugate transpose map, is a C*-algebra.

Definition 2.5. [3] A *real C*-algebra* A is a C*-algebra over $\mathbb{R}$ with the added axiom that for all $x \in A$, $x^*x + 1$ is invertible in the unitisation.

Even though it seems natural to view real C*-algebras as real Banach algebras, one can also view them as *real structures* on complex C*-algebras in the following respect: we have a real structure $\tau : A \to A$ that satisfies

$$\tau(a + \lambda b) = \tau(a) + \lambda\tau(b)$$

$$\tau(ab) = \tau(b)\tau(a)$$

$$\tau(a^*) = \tau(a)^*$$

$$\tau^2 = id_A$$

So τ is a period 2, *-antiautomorphism that acts on a complex C*-algebra. We associate a real C*-algebra A with $A_\tau = \{a \in A | \tau(a) = a^*\}$ [3].

Example 2.6. [3] The transpose map on $M_n(\mathbb{C})$ is a real structure. In this case, $A_\tau = M_n(\mathbb{R})$.

Example 2.7. [3] The sharp map (denoted $\sharp$) on $M_2(\mathbb{C})$ is another real structure. It acts in the following way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sharp = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Here, $A_\tau = \mathbb{H}$, the quaternions.

In Chapter 4, we discuss a commuting real structure that arises from Bratteli diagrams.

2.2 Projection Operators and the Semigroup $\mathcal{D}(A)$

In this subsection, we follow Rørdam, Larsen, and Laustsen's discussion in [16] regarding semigroups of projections as we begin to construct the K^0 group.

Definition 2.8. Let A be a (complex) C*-algebra. An element $p \in A$ is a projection if $p = p^2 = p^*$. We let $\mathcal{P}(A)$ denote the set of projections in A.

We endow $\mathcal{P}(A)$ with the following equivalence relation: $p \sim q$ if there exists a $v \in A$ such that $p = v^*v$ and $q = vv^*$. We call this *Murray-von Neumann equivalence*. We want to generalize Murray-von Neumann equivalence for projections of different sized matrix algebras. Let A be a C*-algebra and n a positive integer. Define

$$\mathcal{P}_n(A) = \mathcal{P}(M_n(A)) \text{ and } \mathcal{P}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(A)$$

Let p be a projection in $\mathcal{P}_n(A)$ and q a projection in $\mathcal{P}_m(A)$. Define the relation $\sim_0$ on $\mathcal{P}_\infty(A)$ such that p$\sim_0$ q if and only if there is a $v \in M_{m,n}(A)$ with $p = v^*v$ and $q = vv^*$. Here, v^* is the conjugate transpose matrix of v, and $M_{m,n}(A)$ is the set of $m \times n$ matrices with entries from A. $\sim_0$ is an equivalence relation on $\mathcal{P}_\infty(A)$. It generalizes Murray-von Neumann equivalence in the sense that if p and $q \in \mathcal{P}_n(A)$, p$\sim_0$ q if and only if p and q are Murray-von Neumann equivalent.

We equip $\mathcal{P}_\infty(A)$ with the binary operation $\oplus$ which we define to be

$$p \oplus q = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

where $p \oplus q \in \mathcal{P}_{n+m}(A)$.

Definition 2.9. Let $(\mathcal{P}_\infty(A), \sim_0, \oplus)$ be as above. We define

$$\mathcal{D}(A) = \mathcal{P}_\infty(A)/\sim_0.$$

In $\mathcal{D}(A)$, we let $[p]_\mathcal{D}$ denote the equivalence class of p. We define addition on $\mathcal{D}(A)$ in the following way:

$$[p]_\mathcal{D} + [q]_\mathcal{D} = [p \oplus q]_\mathcal{D}, \qquad p, q \in \mathcal{P}_\infty(A).$$

Under addition, $\mathcal{D}(A)$ is an abelian semigroup. In the following subsection, we use the Grothendieck construction to generalize how a group is constructed from a semigroup.

2.3 The Grothendieck Construction

In the Grothendieck construction, we generalize the construction of the group $\mathbb{Z}$ from the semigroup $\mathbb{N}$. We follow Rotman's construction in [17].

Let $(S, +, 0)$ be an abelian semigroup. We give S the *cancellation property*, which means that if $x, y, z \in S$ and $x + z = y + z$, it follows that $x = y$. On $S \times S$, we define the element (x, y) to be $x - y$. In this sense, we may define an equivalence relation $\sim$ on $S \times S$ by $(x, y) \sim (a, b)$ if and only if $x + b = y + a$. Let $[(x, y)]$ denote the equivalence class of (x, y). We define the addition

$$[(x, y)] \oplus [(a, b)] = [(x + a, y + b)].$$

Let $(x, y) = (x', y')$ and $(a, b) = (a', b')$. We have that

$$[(x + a, y + b)] \sim [(x + a, y + b)]$$

$$\iff x + b + x' + b' = y + a + y' + a',$$

which holds. Thus our addition is well-defined. We have

$$[(x, x)] \sim [(0, 0)] = id$$
$$[(x, y)] \oplus [(y, x)] = [(y, y)] \sim [(0, 0)]$$

Since addition is associative, this shows that $(S \times S)/\sim$ is an abelian group. We call $G(S) = (S \times S)/\sim$ the Grothendieck group of the semigroup S. We get an injective map γ_S with cancellation:

$$\gamma_S : S \to G(S) : m \mapsto [(m, 0)]$$

We call γ_S the *Grothendieck map*.

Example 2.10. Consider the semigroup $(\mathbb{N}, +)$. In the Grothendieck construction, we have the following equivalence relation $\sim$ between the formal differences of natural numbers:

$$n - m \sim n' - m' \iff n + m' = n' + m$$

Define for all $n \in \mathbb{N}$:

$$\begin{cases} n := [n - 0] \\ -n := [0 - n] \end{cases}$$

This defines the group $\mathbb{Z}$ under addition.

2.4 The Standard Picture of K^0

We now have the necessary framework to introduce the K^0 group for unital C*-algebras. We return to the discussion in [16].

Definition 2.11. Let A be a unital C*-algebra and let $\mathcal{D}(A)$ be the semigroup under addition as defined above. We define $K^0(A)$ by

$$K^0(A) = G(\mathcal{D}(A)).$$

In other words, $K^0(A)$ is the Grothendieck group of $\mathcal{D}(A)$. This definition for K^0 works only if the semigroup S in the Grothendieck construction has the cancellation property. There is a more general definition for K^0 without using cancellation, but the one above will suffice for our purposes.

We also define the map $[\cdot]$: $\mathcal{P}_\infty(A) \to K^0(A)$ by

$$[p]_0 = \gamma([p]_{\mathcal{D}}) \in K^0(A), \qquad p \in \mathcal{P}_\infty(A),$$

where $\gamma : \mathcal{D}(A) \to K^0(A)$ is the Grothendieck map.

The following proposition provides a more useful description of K^0.

Proposition 2.12. If A is a unital C*-algebra, then

$$K^0(A) = \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\}$$

$$= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A),\ n \in \mathbb{N}\}$$

Example 2.13. Using the picture of K^0 given in Proposition 2.12, it can be shown that $K^0(M_n(\mathbb{C})) \cong \mathbb{Z}$.

In the context of Cantor minimal systems, we view the Grothendieck group as a dimension group.

Definition 2.14. [10]. A partially ordered group $(G, +)$ is a dimension group if it satisfies the following properties:

- $(G, +)$ is directed, that is, every subset of G has an upper bound, or equivalently, a lower bound in G
- $(G, +)$ is unperforated, that is, for every $g \in G$, if $ng \geq 0$, then $g \geq 0$, where n is a positive integer
- $(G, +)$ satisfies the *Riesz interpolation property*: Let $a, b, c, d \in G$. If

$$\begin{array}{ccc} a & & c \\ & \leq & \\ b & & d \end{array}$$

then there exists an $h \in G$ such that

$$\begin{array}{ccccc} a & & & & c \\ & \leq & h & \leq & \\ b & & & & d \end{array}$$

2.5 The Classification of Cantor Minimal Systems under K^0

Definition 2.15. Two dynamical (X, T) and (Y, S) systems are said to be *conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ T = S \circ h$. They are said to be *flip conjugate* if (X, T) is conjugate to either (Y, S) or (Y, S^{-1}).

Definition 2.16. Two Cantor minimal systems (X, T) and (Y, S) are said to be *orbit equivalent* if there exists a homeomorphism $F : X \to Y$ such that for all $x \in X$, $F(\mathcal{O}_T(x)) = \mathcal{O}_S(F(x))$.

Remark 2.17. [7] One can check that flip conjugacy, and hence conjugacy, imply orbit equivalence.

With Cantor minimal systems (X, T) and (Y, S), there are two maps, $n : X \to \mathbb{Z}$ and $m : Y \to \mathbb{Z}$ that arise naturally from orbit equivalence: for each $x \in X$, $F(T(x)) = S^{n(x)}(F(x))$, and similarly, $F(T^{m(x)}(x)) = S(F(x))$. From minimality, it follows that n and m are uniquely defined. We call these maps *orbit cocycles*.

Definition 2.18. Two Cantor minimal systems (X, T) and (Y, S) are said to be *strongly orbit equivalent* if they are orbit equivalent and the orbit cocycles that arise from them each have at most one point of discontinuity.

Theorem 2.19. Strong orbit equivalence is in fact an equivalence relation.

Proof. Let (X, T) be a Cantor minimal system and take $F(x) = x$, the identity map. This is a homeomorphism. We have $F(\mathcal{O}_T(x)) = \mathcal{O}_T(F(x))$ if and only if $\mathcal{O}_T(x) = \mathcal{O}_T(x)$, so X is orbit equivalent to itself. We have $F(T(x)) = T^{n(x)}(F(x))$ if and only if $T(x) = T^{n(x)}(x)$, so the cocycle is $n(x) = 1$. n is everywhere continuous, so X is strongly orbit equivalent to itself. So the reflexive property holds. Next, assume (X, T) and (Y, S) are strongly orbit equivalent by $F : X \to Y$ with cocycles $n : X \to \mathbb{Z}$ and $m : Y \to \mathbb{Z}$. Then $F^{-1} : Y \to X$ induces strong orbit equivalence on (Y, S) and (X, T) with the same cocycles. Thus the symmetric property holds. Lastly, suppose (X, T) and (Y, S) are strongly orbit equivalent by $F : X \to Y$ with cocycles $n : X \to \mathbb{Z}$ and $m : Y \to \mathbb{Z}$, and suppose (Y, S) and (V, U) are strongly orbit equivalent by $H : Y \to V$ with cocycles $a : Y \to \mathbb{Z}$ and $b : V \to \mathbb{Z}$. Then the homeomorphism $F \circ H$ induces strong orbit equivalence on (X, T) and (V, U) with cocycles n and b as above. Thus transitivity holds.

□

In 1995, Giordano, Putnam, and Skau proved that K_0, as a dimension group, classifies Cantor minimal systems under orbit equivalence and strong orbit equivalence up to isomorphism.

Theorem 2.20 ([7], Theorem 2.2)**.** Let (X, T) and (Y, S) be two Cantor minimal systems. The following are equivalent:

i) (X, T) and (Y, S) are strongly orbit equivalent

ii) $(K^0(X, T), K^0(X, T)^+, [1_X]) \cong (K^0(Y, S), K^0(Y, S)^+, [1_Y])$

Here, $K^0(X, T)^+$ denotes the equivalence class of nonnegative continuous integer valued functions on X. $[1_X]$ is the equivalence class of the constant function on X. Together, the system $(K^0(X, T), K^0(X, T)^+, [1_X])$ forms a simple non-cyclic dimension group [5].

We define the infinitesimal subgroup $Inf(G)$ of a partially ordered group G (with order unit u) to be the set of all elements g such that for all $0 < \epsilon \in \mathbb{Q}$, we have $-\epsilon u < g < \epsilon u$. We let $K^0_m(X, T) = K^0(X, T)/Inf(K^0(X, T))$ [5].

Giordano, Putnam, and Skau have a similar result for orbit equivalence.

Theorem 2.21 ([7], Theorem 2.1)**.** Let (X, T) and (Y, S) be two Cantor minimal systems. The following are equivalent:

i) (X, T) and (Y, S) are orbit equivalent

ii) $(K_m^0(X, T), K_m^0(X, T)^+, [1_X]) \cong (K_m^0(Y, S), K_m^0(Y, S)^+, [1_Y])$

3 Bratteli Diagrams and the Bratteli-Vershik Model

In 1972, Ola Bratteli introduced the notion of Bratteli diagrams as a combinatorial model for approximately finite dimensional algebras [1]. In this chapter, we introduce Bratteli diagrams in the context of matrix algebras. We equip each Brattli diagram with a proper ordering. The chapter culminates in the remarkable result of Herman, Putnam, and Skau that relates Cantor minimal systems and Bratteli diagrams [12].

3.1 Bratteli Diagrams

Definition 3.1. [6] A *Bratteli diagram* (V, E) consists of a set of vertices $V = \sqcup_{n=0}^{\infty} V_n$ and a set of edges $E = \sqcup_{n=0}^{\infty} E_n$, with $V_0 = \{v_0\}$ a one point set, and each V_n and E_n is finite. Each Bratteli diagram has a range map $r : E_n \to V_n$ and a source map $s : E_n \to V_{n-1}$ such that if $e \in E_n$ connects $v \in V_{n-1}$ and $u \in V_n$, we write $r(e) = u$ and $s(e) = v$.

We can describe the edges of E_n by a $|V_{n-1}| \times |V_n|$ incidence matrix, where the entry a_{ij} denotes the number of edges connecting $v_i \in V_{n-1}$ and $u_j \in V_n$.

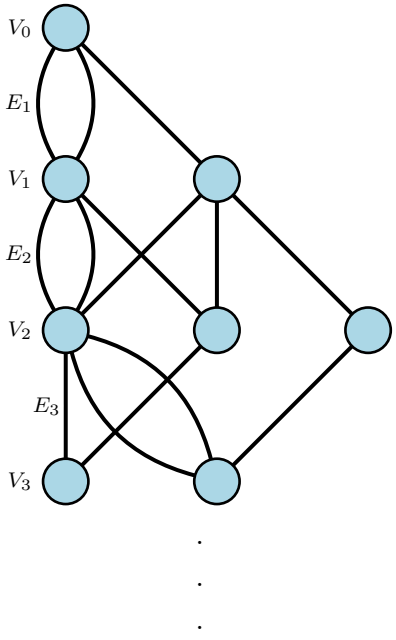

Figure 1: The first stages of a Bratteli diagram (V, E). The incidence matrix for E_1 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, the incidence matrix for E_2 is $\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, and the incidence matrix for E_3 is $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

Let k and l be positive integers with $k < l$. We define E_k^l to be the set of all paths from V_{k-1}

to V_l where there is only one edge e_i at each stage E_n, $k \leq n \leq l$. If A_n denotes the incidence matrix of E_n, we may contract a Bratelli diagram (V, E) to a diagram (V', E') by letting the matrix $A_l A_{l-1} \cdots A_k$ be the incidence matrix of $E_{k'}$ [6]. In this sense, we telescope E_k^l down to $E_{k'}$. We call (V', E') a *telescoping* of (V, E).

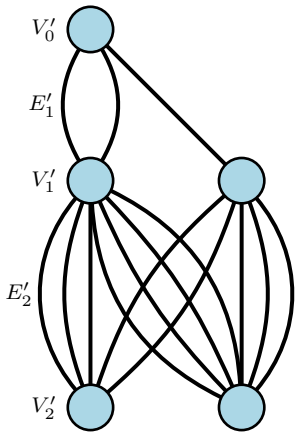

Figure 2: (V', E') is a telescoping of (V, E) where we have telescoped E_2^3. From (V, E), we have $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$. Note that the number of edges connecting $v'_{1_i} \in V_1'$ and $v'_{2_j} \in V_2'$ is the same as the number of paths in (V, E) from $v_{1_i} \in V_1$ to $v_{3_j} \in V_3$.

Definition 3.2. [13] An approximately finite dimensional algebra (*AF-algebra*) A is a C*-algebra containing a sequence of finite dimensional algebras $\bigcup_{n=1}^{\infty} A_n$ such that the closure of $\bigcup_{n=1}^{\infty} A_n$ is equal to A.

Bratteli Diagrams give rise to AF-algebras in the following way [1]. Suppose we have a sequence of finite dimensional C*-algebras

$$\mathbb{C} \cdot 1 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq ... \subseteq A = \bigcup_{n=1}^{\infty} A_n$$

where

$$A_i = M_{n_1^i}(\mathbb{C}) \oplus M_{n_2^i}(\mathbb{C}) \oplus ... \oplus M_{n_{k_j}^i}(\mathbb{C})$$

In a Bratteli diagram, each vertex in row i denotes a matrix algebra, and A_i is the direct summand of those algebras. The edges denote the multiplicity in the embedding from A_i to A_{i+1}. For example, the Bratteli diagram in Figure 1 encodes the embedding

$$\mathbb{C} \cdot 1 \to M_2(\mathbb{C}) \oplus \mathbb{C} \to M_5(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \to M_8(\mathbb{C}) \oplus M_{11}(\mathbb{C})$$

3.2 Ordered Bratteli Diagrams

For the rest of Chapter 3, we follow Giordano's discussion in [6], p. 439-441.

Definition 3.3. We say a Bratteli diagram (V, E) is *simple* it it admits a telescoping such that the resulting incidence matrices have strictly positive entries. Equivalently, (V, E) is simple if at any level V_n, there exists an $m > n$ such that there is at least one path from any vertex $v \in V_n$ to any vertex $u \in V_m$.

Definition 3.4. Given a Bratteli diagram (V, E), its infinite path space is the space of all paths with exactly one edge at each level. We define it to be $X_{(V,E)} = \{(e_1, e_2, ...)|e_i \in E_i,\ r(e_i) = s(e_{i+1})\ \forall i \geq 1\}$.

We may equip $X_{(V,E)}$ with the relative topology, which means that, roughly speaking, two paths that are identical on a long initial stretch are considered to be close. If we consider a finite path $p = (e_1, e_2, ..., e_n)$, we define its cylinder set $U(p)$ to be the collection of infinite paths that agree with p on their initial stretches. More formally, $U(p) = \{(f_1, f_2, ...) \in X_{(V,E)}|f_i = e_i,\ i = 1, 2, ..., n\}$. The cylinder sets form a basis for the relative topology. They are also closed and open, which means that $X_{(V,E)}$ is compact and totally disconnected. We may also define a metric on $X_{(V,E)}$ where $d(x, y) = 1/n$ whenever edges x and y agree up to but not including level n. If (V, E) is simple, then $X_{(V,E)}$ has no isolated points. So, with the results from above, this makes $X_{(V,E)}$ a Cantor set.

Definition 3.5. We may equip a Bratteli diagram (V, E) with a partial order $\geq$ such that two edges e and f are comparable in the order if and only if they connect to the same vertex (ie. $r(e) = r(f)$). We call $(V, E, \geq)$ *an ordered Bratteli diagram.*

In an ordered Bratteli diagram $(V, E, \geq)$, we equip the set of paths in $E_n \circ E_{n+1} \circ ... \circ E_k$ that have the same range with the lexicographic ordering as follows: $(e_n, e_{n+1}, ..., e_k) > (f_n, f_{n+1}, ..., f_k)$, if and only if, for some $l \leq k$, $e_i = f_i$ for $n \leq i \leq l - 1$, and $e_l > f_l$ in the partial order.

If we equip a simple Bratteli diagram with a partial ordering, we say it is properly ordered if there exist unique minimal and maximal path. That is, there exist $x_{min} = (e_1, e_2, ...)$ and $x_{max} = (f_1, f_2, ...) \in X_{(V,E)}$ such that each e_i is a minimal edge and each f_i is a maximal edge. Any simple Bratteli diagram can be properly ordered.

3.3 The Bratteli-Vershik Model

Let $X_{(V,E)}$ be the path space associated with a properly ordered Bratteli diagram. As we have previously noted, $X_{(V,E)}$ is a Cantor set. We define the lexicographic map on $X_{(V,E)}$, $\varphi_{(V,E)}$, in the following way. Let $x = (e_1, e_2, ...) \in X_{(V,E)}$, $x \neq x_{max}$, and let k be the smallest integer such that e_k is not maximal. Let f_k be the successor of e_k in the ordering. Then, $\varphi_{(V,E)}(x) = (f_1, ...f_{k-1}, f_k, e_{k+1}, e_{k+2}, ...)$, where $(f_1, ..., f_{k-1})$ is the initial string of the unique minimal path

such that $r(f_{k-1}) = s(f_k)$. Thus, x and $\varphi_{(V,E)}(x)$ agree from level $k+1$ onwards. We define $\varphi_{(V,E)}(x)$ to "wrap around" so that $\varphi_{(V,E)}(x_{max}) = x_{min}$. Notice how $\varphi_{(V,E)}$ is just a generalization of the odometer in Example 1.23. $\varphi_{(V,E)}$ is a minimal homeomorphism, making $(X_{(V,E)}, \varphi_{(V,E)})$ a Cantor minimal system. We call $\varphi_{(V,E)}$ the *Vershik map* and $(X_{(V,E)}, \varphi_{(V,E)})$ a *Bratteli-Vershik system*.

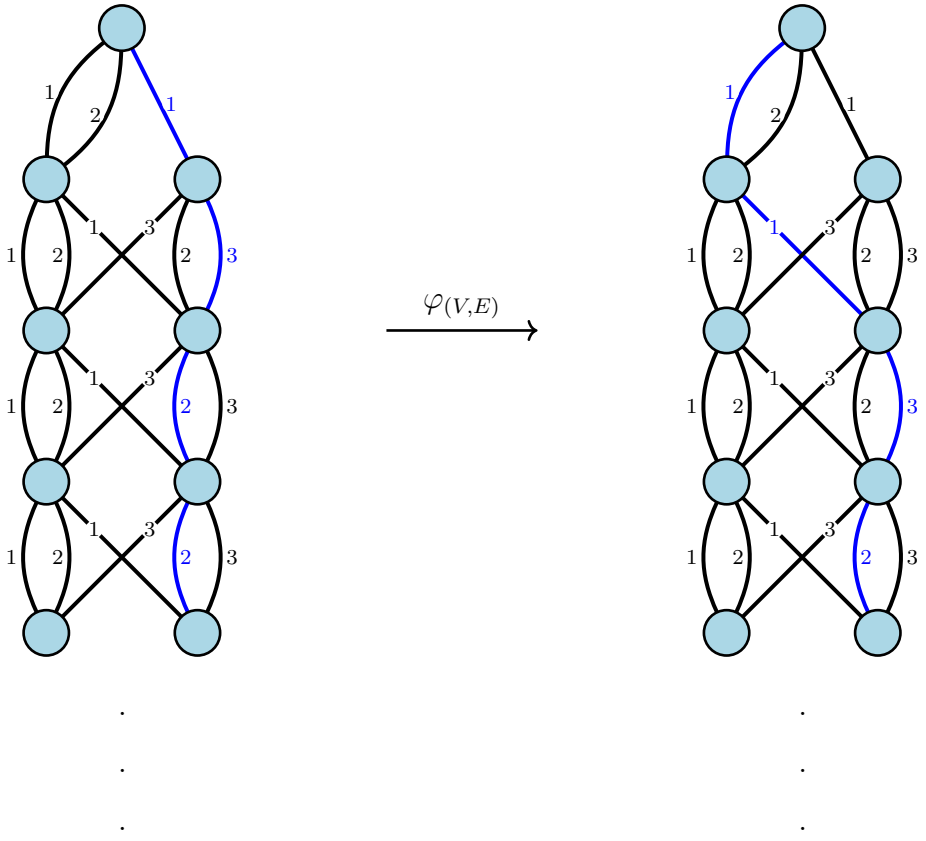

Figure 3: [8] An example of the Vershik map acting on a path in a properly ordered stationary Bratteli diagram. The coloured paths indicate x and $\varphi_{(V,E)}(x)$.

Herman, Putnam, and Skau provide us with the remarkable result that any Cantor minimal system can be represented by the construction in this chapter.

Theorem 3.6. [12]. For every Cantor minimal system (X, φ), there exists a properly ordered Bratteli diagram $(V, E, \geq)$ such that the corresponding Bratteli-Vershik system $(X_{(V,E)}, \varphi_{(V,E)})$ is conjugate to (X, φ).

4 Bratteli-Vershik Systems as Models of Cantor Minimal Systems

In this chapter, we discuss another way of viewing how minimal actions act on Bratteli diagrams, and we conclude by constructing a Bratteli-Vershik system that yields a commuting $\mathbb{Z}/2\mathbb{Z}$ action.

4.1 An Equivalent Definition of Minimality

Consider a properly ordered Bratteli diagram $(V, E, \geq)$. As discussed previously, the infinite path space, $X_{(V,E)}$, is closed as a subset of $V_0 \times V_1 \times ...$, and is therefore compact. Recall that an action T on $X_{(V,E)}$ is minimal if the orbit $\mathcal{O}_T(x)$ is dense for all $x \in X_{(V,E)}$. Equivalently, T is minimal if $(\overline{\mathcal{O}_T(x)})^{\complement}$ is empty.

In the product topology, the n-neighbourhood of a path x is the set of all paths who's first n edges agree with x. In other words, the n-neighbourhood of $x = (e_1, e_2, ...)$ is the cylinder set $U(p)$ where $p = (e_1, ..., e_n)$. These cylinder sets are the basic neighbourhoods in the relative topology as a subset of the whole product. Now let y and z be arbitrary paths. $\mathcal{O}_T(y)$ is dense if for all neighbourhoods $U(p)$ of z, there exists an $m \in \mathbb{N}^+$ such that $T^m(y) \in U(p)$. This gives us an equivalent notion of minimality on a properly ordered Bratteli diagram.

In a Bratteli-Vershik system $(X_{(V,E)}, \varphi_{(V,E)})$, $\varphi_{(V,E)}$ permutes paths in a cyclic way. That is, it will eventually permute a path x into any n-neighbourhood of another path x_0. This makes it easy to see that $(X_{(V,E)}, \varphi_{(V,E)})$ has dense orbits and shows that $\varphi_{(V,E)}$ is minimal.

4.2 An Example and some Results from K-theory

Example 4.1. Consider a properly ordered Bratteli diagram $(V, E, \geq)$ where $|V_i| = 1$ for all $i \in \mathbb{N}$. Let ψ be a $\mathbb{Z}$-action on $X_{(V,E)}$ by translation on each row: $\psi(x) = \psi((e_1, e_2, ...)) = (e_1', e_2', ...)$, where e_m' is the successor of e_m in row E_i. We define ψ to have wrap around so that if e_m is the maximal edge in E_i, then e_m' is the minimal edge in E_i. We can see that ψ acts cyclically on E_i. That is, if $|E_i| = n_i$, then $\psi^k(e_m) = \psi^l(e_m)$ if and only if $k \equiv r \pmod{n_i}$. So, ψ is a $\mathbb{Z}/n_i\mathbb{Z}$ action on E_i. We want a way to assure that ψ is a minimal action on $X_{(V,E)}$. From section 4.1, this means ψ must eventually permute a finite path to any other finite path. Since ψ is a $\mathbb{Z}/n_i\mathbb{Z}$ action on each E_i, by the Chinese Remainder Theorem, the minimality condition is only met if we construct $(V, E, \geq)$ so that all the n_is are all coprime. Let us also allow one of these n_i to be even: say $n_j = 2^{k_j}$. We get a $\mathbb{Z}/2\mathbb{Z}$ action on E_j since $\mathbb{Z}/2\mathbb{Z}$ is a cyclic subgroup of $\mathbb{Z}/2^{k_j}\mathbb{Z}$. For instance, if $n_j = 8$, then our $\mathbb{Z}/2\mathbb{Z}$ action is $\psi^4(x)$. Also, ψ maps (x) to the same path whether the $\mathbb{Z}/2\mathbb{Z}$ is applied before or after ψ acts on the other rows, which makes the action commutative. So by constructing $(V, E, \geq)$ in this was, ψ is a minimal action and gives us a commuting $\mathbb{Z}/2\mathbb{Z}$ action.

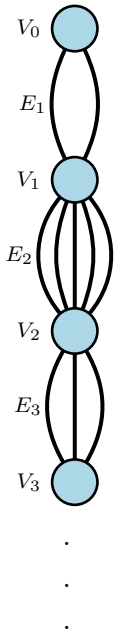

Figure 4: A Bratteli diagram that meets the criteria of Example 4.1 on its first three stages. ψ is a minimal action since $|E_1| = 2$, $|E_2| = 5$, and $|E_3| = 3$ are all coprime. We get a commuting $\mathbb{Z}/2\mathbb{Z}$ as described above.

We associate the infinite path space $X_{(V,E)}$ with the C*-algebra $A = C(X_{(V,E)})$, ie. the set of continuous complex-valued functions on $X_{(V,E)}$ [15]. Since our $\mathbb{Z}/2\mathbb{Z}$ action is commutative, the real structure on this algebra is the same as the action itself. We have $A_\tau = \{x \in A \mid \tau(x) = x^*\} = \{f \in C(X_{(V,E)}) \mid f(\tau(x)) = \overline{f(x)}\}$. Looking at the K-theory of $C(X_{(V,E)})$, we have that $K^0(C(X))$ yields the set of continuous valued functions on $X_{(V,E)}$ [15]. In [15], Putnam also discusses that $\mathrm{C}(X_{(V,E)})$ generates the crossed product algebra $C(X_{(V,E)}) \rtimes_\varphi \mathbb{Z}$ with a unitary operator u that satisfies $ufu^* = \varphi(f)$ for all $f \in C(X_{(V,E)})$. Looking at the K-theory of this algebra, we have $K^0(C(X_{(V,E)}) \rtimes_\varphi \mathbb{Z}) \cong C(X, \mathbb{Z})/Im(id - \varphi_*)$, where $Im(id - \varphi_*)$ is the kernel of the inclusion of $C(X_{(V,E)})$ into $C(X_{(V,E)}) \rtimes_\varphi \mathbb{Z}$ [15]. An area of further investigation is how the K-theory of the real structures of the crossed-product algebra adds to the picture.

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